Section 2.0: Mathematical Programming

The next five chapters in the text focus on mathematical programming. The father of mathematical programming is George Dantzig. Between 1947 and 1949, Dantzig developed the basic concepts used for framing and solving linear programming problems. During WWII, he worked on developing various plans which the military called "programs." After the war he was challenged to find an efficient way to develop and solve these programs.

Dantzig recognized that these programs could be formulated as a system of linear inequalities. Next, he introduced the concept of a goal. At that time, goals usually meant rules of thumb for carrying out a goal. For example, a navy admiral might have said, "Our goal is to win the war, and we can do that by building more battleships." Dantzig was the first to express the selection of a plan to reach a goal as a mathematical function. Today it is called the *objective function*.

All of this work would not have had much practical value without a way to solve the problem. Dantzig found an efficient method called the simplex method. This mathematical technique finds the optimal solution to a set of linear inequalities that maximizes (profit) or minimizes (cost) an objective function.

Economists were excited by these developments. Several attended an early conference on linear programming and the simplex method called "Activity Analysis of Production and Allocation." Some of them later won Nobel prizes in economics for their work. They were able to model fundamental economic principles using linear programming.

The first problem Dantzig solved was a minimum cost diet problem. The problem involved the solution of nine inequalities (nutrition requirements) with seventy-seven decision variables (sources of nutrition). The National Bureau of Standards supervised the solution process. It took the equivalent of one man working 120 days using a hand-operated desk calculator to solve the problem. Nowadays, a standard personal computer could solve this problem in less than one second. Excel spreadsheet software includes a standard "add-in" called "solver", a tool for solving linear programming problems.

Mainframe computers became available in the 1950s and grew more and more powerful. This allowed many industries, such as the petroleum and chemical industries, to use the simplex method to solve practical problems. The field of linear programming grew very fast. This led to the development of non-linear programming, in which inequalities and/or the objective function are not linear functions. Another extension is called integer programming, in which the variables can only have integer values. Together, linear, non-linear and integer programming are called **mathematical programming**.

2.0.1 An Introductory Problem

In order to get a feel for mathematical programming, this chapter begins with a problem that has a concrete model. This model can be built from Lego pieces. When a mathematical model of a real world situation is constructed in symbolic form, it is often helpful to construct a physical or visual model at the same time. The role of the latter model is to help the model builder to understand the real-world situation as well as its mathematical model.

The Problem

A certain furniture company makes only two products: tables and chairs. The manufacturing of tables and chairs can be modeled using Lego pieces. To make a table requires two large and two small pieces, and a chair requires one large and two small pieces. Figure 2.0.1 shows a table and a chair made from Legos.

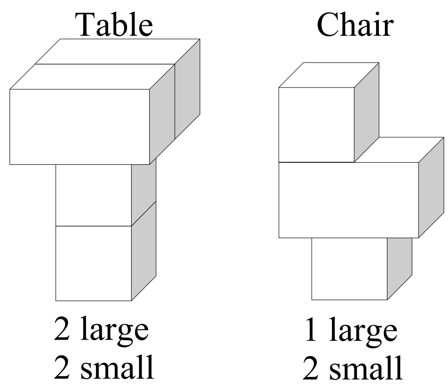

Figure 2.0.1: A Lego table and chair

If the resources needed to build tables and chairs were unlimited, the company would just manufacture as many of each as it thought it could sell. In the real world, however, resources are not unlimited. Suppose that the company can only obtain six large and eight small pieces per day. Figure 2.0.2 shows these limited resources.

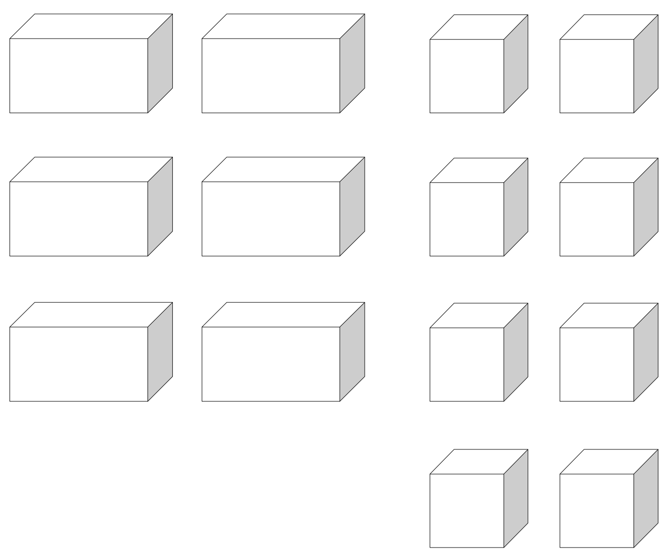

Figure 2.0.2: The furniture company's limited resources

The profit from each table is \$16, and the profit from each chair is \$10. The production manager wants to find the rate of production of tables and chairs per day that earns the most profit. **Production rate** refers to the number of tables and chairs this company can produce per day.

Q1. What do you think the production rates should be in order to generate the most profit?

Q2. Does the number of table and chairs produced each day have to be an integer value?

Q3. Using only eight small and six large Legos, build a physical model of this problem. If Legos are unavailable, draw pictures to explore some possibilities. Create several combinations of tables and chairs this company could make using your model.

Solving the Problem

There are many possible product mixes this company could make. A **product mix** is a combination of each product being manufactured. The various product mixes could be explored using the Lego model.

First, the company could begin by making as many tables as possible since the profit from a table is much greater than the profit from a chair. Each table requires two large pieces and two small pieces. There are only six large and eight small pieces available. Therefore, only three tables can be built. This generates 3(\$16) = \$48 profit. There are two small pieces left over, but nothing can be built from them. Thus, \$48 is the total profit if three tables (and no chairs) are built.

Three tables and zero chairs was one possible product mix. There could be other production rates that generate more profit.

No more than three tables could be made due to the limited resources available, and making three tables yielded a profit of \$48. Now, suppose two tables are made. Manufacturing two tables uses four large and four small pieces. Now there are two large and four small pieces left over. These are just enough resources to build two chairs. The profit on two tables and two chairs would be 2(\$16) + 2(\$10) = \$52. This is more profit than building three tables. However, the production manager wonders, "Is \$52 the greatest profit possible? Is there another product mix that could generate more profit?"

Q4. In a Table 2.0.1, record other combinations of tables and chairs the company could produce. For each combination, write the production rate of tables, the production rate of chairs, and the profit for each possibility.

Production Rate of Tables	Production Rate of Chairs	Total Profit

Table 2.0.1: Exploring the total profit for each combination of tables and chairs

Q5. Which production rates generate the most profit?

Q6. Did any product mix yield a profit greater than \$52?

It is impossible to find the total profit for every product mix because there are infinitely many possibilities. However, most likely no one in the class found a profit greater than $52. In the next section, you will learn how to know for certain you found the product mix with the greatest profit.

Notice that in Table 2.0.1 you used a set of similar equations to compute the profit for each possibility. These equations are the basis for the **objective function**. The two production rates *varied* across each possible product mix, and exploring these variations allows a *decision* about production to be made. Therefore, the production rates for tables and chairs are known as the **decision variables** for this problem.

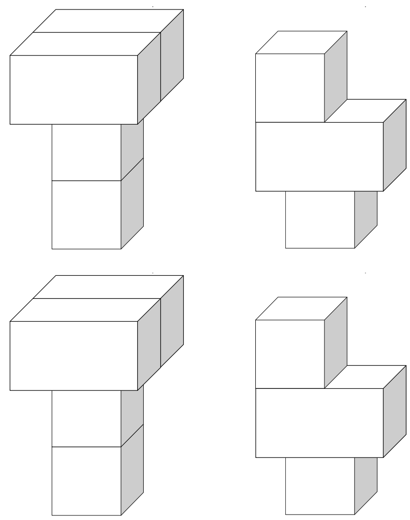

Figure 2.0.3: Two tables and two chairs yield the most profit

Because the profit has been optimized, the solution in Figure 2.0.3 is called the **optimal solution**. Besides the optimal solution, there are many other possible solutions. Although they are not optimal, each possible solution is still a **feasible** solution. Building four tables is an example of an **infeasible** solution.

Q7. Why is building four tables an example of an infeasible solution?

Q8. Give another example of an infeasible solution.

Stepping Beyond the Solution

Operations researchers understand that there is more to their work than merely finding solutions to problems. Once a solution is found, it must be interpreted. One sort of interpretation is called **sensitivity analysis**. Sensitivity analysis involves exploring how sensitive the solution is to changes in the parameters of the problem. For example, in the Lego problem above, one of the parameters of the problem is the availability of large pieces.

Q9. Would it make a difference if *seven* large pieces were available instead of six (there are still eight small pieces)? If so, what is the new optimal solution, and how much profit does it generate?

Q10. Would it make a difference if *nine* small pieces were available instead of eight (there are still six large pieces)? If so, what is the new optimal solution, and how much profit does it generate?

Q11. Would it make a difference if *seven* large pieces and *nine* small pieces were available? If so, what is the new optimal solution, and how much profit does it generate?

Growing the Problem

Suppose now that the furniture company has decided to dramatically expand production. Now it is able to obtain 27 small and 18 large Lego pieces per day. The profit on tables and chairs remains the same.

Q12. What should the daily production rates be in order to maximize profit?

Q13. Would it make a difference if 19 large pieces were available instead of 18 (there are still 27 small pieces)? If so, what is the new optimal solution, and how much profit does it generate?

Q14. Would it make a difference if 28 small pieces were available instead of 27 (there are still 18 large pieces)? If so, what is the new optimal solution, and how much profit does it generate?

Q15. Was this new problem easier or more difficult to solve than the original? Why?

