Section 5.1: Political Advertising

Cynthia Brown is running for the office of governor in Michigan. She is considering purchasing ads in several newspapers costing a total of $80,000 per day or TV ads costing $60,000 per day. She decides she does not want to order more than one ad per day on any medium (i.e., no more than seven newspaper ads per week and no more than seven TV ads per week). She wonders how many of each type of ad she could purchase using $600,000.

Q1. Suppose Cynthia Brown lets x_1 represent the number of newspaper ads purchased and x_2 represent the number of TV ads purchased. Explain why the inequality $80x_1 + 60x_2 \leq 600$ models this situation.

Q2. Why is 80 used instead of 80,000, 60 instead of 60,000, and 600 instead of 600,000?

First, Cynthia Brown wants to know how many of each type of ad she can purchase for *exactly* $600,000. Thus, she uses algebra to find a solution to the equation $80x_1 + 60x_2 = 600$. She starts by choosing any number of newspaper ads and solving for the number of TV ads. For example, if she chooses 5 newspaper ads (i.e., she lets $x_1 = 5$), then she has $80(5) + 60x_2 = 600$. Then she solves for x_2:

$$\begin{aligned} 80(5) + 60x_2 &= 600 \\ 400 + 60x_2 &= 600 \\ 60x_2 &= 200 \\ x_2 &= \frac{200}{60} = 3\frac{1}{3} \end{aligned}$$

Q3. Why is this solution not feasible?

Q4. Does this one infeasible solution mean that there are no solutions at all?

Q5. Find a feasible solution to this equation.

The key to understanding this problem is identifying the need for the solutions to be integers. The candidate must purchase a whole number of ads; she cannot purchase half an ad. Although newspaper ads may be purchased in fractions of a page, not all fractions are possible. For example, a half-page ad could be purchased, but not a 0.345-page ad. Thus, the decision variable is not continuous. If Cynthia Brown wanted to include the possibility of half page and quarter page ads, she would need to introduce another integer decision variable for each sized ad. She chose not to increase the problem size by adding these decision variables.

A linear equation that has two variables with integer coefficients in which we seek integer solutions is called a *linear Diophantine equation*. Requiring an integer solution actually makes the problem more complicated. Here, the goal is to find an easy criterion to test whether an integer solution exists.

Consider again the equation $80x_1 + 60x_2 = 600$. If Cynthia Brown decides to purchase 3 newspaper ads and she lets $x_1 = 3$, then

$$\begin{aligned} 80(3)+60x_2 &= 600 \\ 240+60x_2 &= 600 \\ 60x_2 &= 360 \\ x_2 &= 6 \end{aligned}$$

So the equation has the solution $x_1 = 3$ and $x_2 = 6$, and purchasing 3 newspaper ads and 6 TV ads is feasible. Now, suppose the candidate decides that she can afford to spend $650,000 on advertising.

Q6. How does that change the equation?

Q7. Do you think there is an integer solution to this equation? Why or why not?

Cynthia Brown wonders under what circumstances there will be an integer solution. She ponders to herself, "Is it possible to determine why $80x_1 + 60x_2 = 600$ has an integer solution while $80x_1 + 60x_2 = 650$ does not?"

Since the left hand sides of the two equations are identical, the key lies in the value on the right hand side and its relationship to the left hand side. Cynthia notices that the left hand side of the equation can be simplified by dividing out their greatest common divisor.

Q8. What is the greatest common divisor of the coefficients of x_1 and x_2?

Q9. Divide both sides of the equation $80x_1 + 60x_2 = 600$ by the greatest common divisor found in Q8. Then divide both sides of the equation $80x_1 + 60x_2 = 650$ by the greatest common divisor found in Q8.

Q10. What do you notice about the right hand sides of the two equations in Q9?

The observation just made is the pivotal requirement for the existence of an integer solution to a linear Diophantine equation.

There is an integer solution to a linear Diophantine equation *if and only if* the constant term is divisible by the greatest common divisor (GCD) of the coefficients of x_1 and x_2.

Q11. If x_1 and x_2 are both integers, what does that say about the number on the left hand side of the equation?

Q12. How does this observation about the left hand side of the equation compare with the right hand side of the equations you found in Q9?

At this point, Cynthia Brown has simplified the original equation, but she has yet to find an actual solution. Once she finds a solution, she can ask many more questions: Will there be more than one integer solution? What is the largest number of ads that could be purchased? What is the least number of ads that could be purchased? Is it possible to purchase the same number of TV ads as newspaper ads?

Restricting the decision variables to take on only integer values transforms the linear programming (LP) problem into an **integer programming** (IP) problem. An integer programming problem is more difficult to solve, because as was just seen, a linear equation does not need to have any integer solutions. Geometrically, that means there might not be any points on the graph of a line with integer coordinates. If such a line happens to be part of the boundary of an integer programming formulation, that might lead to

